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# ANALYTICAL SOLUTIONS FOR STRESSES AND DISPLACEMENTS AROUND TUNNELS DRIVEN IN CROSS-ANISOTROPIC ROCKS

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## SUMMARY

In this paper, closed-form solutions for the stresses and displacements around unlined circular tunnels excavated in an elastic, orthotropic (cross-anisotropic) medium are developed. The effect of both the elastic parameters characterizing the behaviour of the medium and the anisotropy of the initial stress system on the stresses and displacements induced are evaluated. An example of utilizing the theoretical solution for design analysis is given. For convenience of application, design charts are prepared for the determination of stresses and displacements for given values of initial stresses and the elastic parameters. Copyright © 1999 John Wiley & Sons, Ltd.

Key words: tunnel; elasticity; cross-anisotropy; stresses; displacements

## INTRODUCTION

When a tunnel is excavated in a rock mass, the initial state of stress is altered, resulting in changes of positions of the points in the medium relative to one another (i.e. strains and displacements are created). The determination of the state of stress and displacements due to excavation are of importance for applications in rock engineering. The elastic solution for stresses around a circular hole in an infinite isotropic plate subjected to a uniaxial stress was obtained by Kirsch (1898) and reported by many authors (see e.g. Reference 1). Mindlin[2] investigated the more general case of a biaxial stress field. Three cases were studied: (a) case of hydrostatic pressure ($K_0 = 1$), where $K_0$ is the initial stress ratio, (b) case of $K_0 = v/1 - v$, where $v$ is Poisson's ratio, and (c) case of no lateral stress. The effect of proximity of boundary on the stress concentration under horizontal *in situ* stress was given by Mindlin.[3] The elastic solution for the tangential stresses around a hole in an infinite medium under general biaxial stresses is known in the literature as Kirsch solution and has often been employed in design (see e.g. Reference 4).

The elastic solution for the displacement around a pre-existing hole in an infinite isotropic plate is given by Obert and Duvall[4] and Jaeger and Cook.[5] However, this solution is not relevant for tunnel design since it does not represent the actual case of deformation due to *in situ* stress

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relief at the tunnel boundary resulting from the excavation. The elastic solution for the displacement resulting from the relief of *in situ* stress was investigated by Yuen[6] and Pender.[7]

It is evident, however, that for sedimentary rocks such as shales, the deformation behaviour is anisotropic. The effects of anisotropy on stresses and displacements resulting from underground excavations have not been studied in detail. The stress distribution around a circular hole in an infinite orthotropic (cross-anisotropic) plate loaded at infinity from one direction was studied by Green and Taylor.[8,9] However, the solutions for the displacements are not studied in detail in the literature.

In this work, closed-form solutions for the stresses and displacements along the circumference of tunnels driven in cross-anisotropic rocks are derived. An example of utilizing the theoretical solution for design analysis is given. Different factors affecting the resulting stresses and displacements, including the initial stress ratio and the degree of anisotropy, are studied. Design charts have been prepared for the determination of stresses and displacements for given values of initial stress ratio ($K_0$) and the elastic parameters.

## CONSTITUTIVE RELATIONSHIPS

For an elastic isotropic material, only two material parameters are required to represent the stress–strain relationships. However, in the case of cross-anisotropic material, properties are different in the horizontal and vertical directions and five independent parameters are involved in the constitutive relationships. Consider the problem as shown in Figure 1, where a tunnel is driven along the $z$-axis such that axes $x$ and $z$ are horizontal and lie on the plane of isotropy. The cross-section lies on the vertical plane ($x$–$y$ plane) which is the plane of anisotropy. The problem is a plane strain problem where the components $\varepsilon_z$, $\varepsilon_{xz}$ and $\varepsilon_{yz}$ vanish everywhere. Therefore, the constitutive relationships are given as

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_{xy} \end{Bmatrix} = \begin{bmatrix} S_{11} & S_{12} & 0 \\ S_{21} & S_{22} & 0 \\ 0 & 0 & S_{33} \end{bmatrix} \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} \tag{1}$$

where $S_{11}$, $S_{22}$, $S_{33}$, $S_{12}$, and $S_{21}$ are deformation coefficients and related to the material parameters as follows:

$$\begin{aligned} S_{11} &= \frac{1-v_{\mathrm{h}}^2}{E_{\mathrm{h}}}, \qquad S_{22} = \frac{1-v_{\mathrm{hv}}v_{\mathrm{vh}}}{E_{\mathrm{v}}} \\ S_{12} = S_{21} &= -\frac{v_{\mathrm{vh}}(1+v_{\mathrm{h}})}{E_{\mathrm{v}}}, \qquad S_{33} = \frac{1}{G_{\mathrm{vh}}} \end{aligned} \tag{2}$$

where $E_{\mathrm{v}}$ is the elastic modulus in vertical direction, $E_{\mathrm{h}}$ the elastic modulus in horizontal direction, $v_{\mathrm{vh}}$ the Poisson's ratio for effect of vertical stress on horizontal strain, $v_{\mathrm{hv}}$ the Poisson's ratio for effect of horizontal stress on vertical strain, $v_{\mathrm{h}}$ the Poisson's ratio for effect of horizontal stress on horizontal strain and $G_{\mathrm{vh}}$ the independent shear modulus in vertical plane.

From energy considerations,[10,11] the following relationships among the elastic parameters must be satisfied:

$$\frac{E_{\mathrm{v}}}{v_{\mathrm{vh}}} = \frac{E_{\mathrm{h}}}{v_{\mathrm{hv}}} \tag{3}$$

$$1-v_{\mathrm{h}} > 0, \quad 1+v_{\mathrm{h}} > 0, \quad 1-v_{\mathrm{h}}-2v_{\mathrm{hv}}v_{\mathrm{vh}} > 0 \tag{4}$$

 

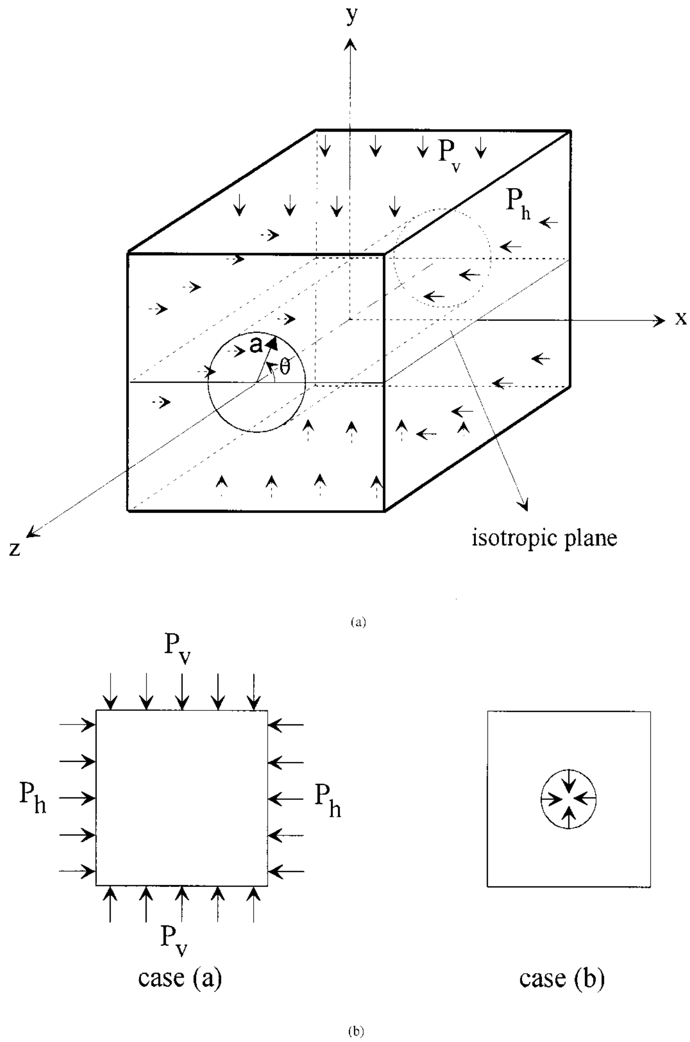


Figure 1. (a) Idealization of the problem and (b) loading cases considered

For the case of isotropy the conditions that

$$E_h = E_v = E, \quad \text{and} \quad \nu_{hv} = \nu_{vh} = \nu_h = \nu \tag{5}$$

are necessary conditions but not sufficient to insure isotropy. A final condition is

$$G_{vh} = G = \frac{E}{2(1+\nu)} \tag{6}$$

## METHOD OF CALCULATIONS

The final stresses due to tunnel excavation in the cross-anisotropic rock may be obtained by the superposition of stresses resulting from two cases (Figure 1): case (a) where the infinite medium loaded by the initial state of stress (before excavating the tunnel), and case (b) of releasing the initial stresses at the boundary of the excavated tunnel. The displacements in the medium due to the excavation of the tunnel result directly from case (b).

The problem of finding the stresses and displacements in an infinite medium with an opening loaded at the circular boundary was reduced by Green and Zerna[12] to the problem of finding two complex potentials $V(z)$ and $W(z)$ satisfying the boundary conditions. The symbol $z$ now represents the complex variable:

$$z = x + \mathrm{i}y \quad \text{in Cartesian co-ordinates, or}$$

$$z = r\mathrm{e}^{\mathrm{i}\theta} \quad \text{in polar co-ordinates}$$

The boundary conditions for stresses at $r = a$, where $a$ is the radius of the tunnel, are represented by $p(a\mathrm{e}^{\mathrm{i}\theta})$ and $q(a\mathrm{e}^{\mathrm{i}\theta})$ for normal and shear streses, respectively. The complex potentials could be represented in a power series as

$$\begin{aligned} V(z) &= \sum_{n=0}^{\infty} (R_n + \mathrm{i}S_n)\frac{a^n}{z^n} \\ W(z) &= \sum_{n=0}^{\infty} (-U_n + \mathrm{i}T_n)\frac{a^n}{z^n} \end{aligned} \tag{7}$$

where

$$\begin{aligned} R_n &= \frac{1}{\pi}\int_0^{2\pi} p(a\mathrm{e}^{\mathrm{i}\theta})\cos n\theta\,\mathrm{d}\theta, & S_n &= \frac{1}{\pi}\int_0^{2\pi} p(a\mathrm{e}^{\mathrm{i}\theta})\sin n\theta\,\mathrm{d}\theta \\ T_n &= \frac{1}{\pi}\int_0^{2\pi} q(a\mathrm{e}^{\mathrm{i}\theta})\cos n\theta\,\mathrm{d}\theta, & U_n &= \frac{1}{\pi}\int_0^{2\pi} q(a\mathrm{e}^{\mathrm{i}\theta})\sin n\theta\,\mathrm{d}\theta \\ R_0 = U_0 &= \frac{1}{\pi}\int_0^{2\pi} p(a\mathrm{e}^{\mathrm{i}\theta})\mathrm{d}\theta, & -S_0 = T_0 &= \frac{1}{2\pi}\int_0^{2\pi} q(a\mathrm{e}^{\mathrm{i}\theta})\,\mathrm{d}\theta \end{aligned} \tag{8}$$

so that $S_n$, $R_n$ and $U_n$, $T_n$ are, respectively, the Fourier coefficients in the expansions of $p(a\mathrm{e}^{\mathrm{i}\theta})$ and $q(a\mathrm{e}^{\mathrm{i}\theta})$ in Fourier series. Therefore,

$$p(a\mathrm{e}^{\mathrm{i}\theta}) = R_0 + \sum_{n=1}^{\infty} (R_n\cos n\theta + S_n\sin n\theta) \tag{9a}$$

$$q(a\mathrm{e}^{\mathrm{i}\theta}) = T_0 + \sum_{n=1}^{\infty} (T_n\cos n\theta + U_n\sin n\theta) \tag{9b}$$

The complex potentials $V(z)$ and $W(z)$ in equation (7) are obtained through equations (8) and (9).

 

The solution for the tangential stresses at the boundary of the hole in terms of these complex potentials is

$$\sigma_\theta = \frac{(1+\gamma_1 e^{-2i\theta})(1+\gamma_2 e^{-2i\theta})V(z) + 2(1-\gamma_1\gamma_2 e^{-4i\theta})W(z)}{2(1-\gamma_1 e^{-2i\theta})(1-\gamma_2 e^{-2i\theta})} + \frac{(1+\bar{\gamma}_1 e^{2i\theta})(1+\bar{\gamma}_2 e^{2i\theta})\bar{V}(\bar{z}) + 2(1-\bar{\gamma}_1\bar{\gamma}_2 e^{4i\theta})\bar{W}(\bar{z})}{2(1-\bar{\gamma}_1 e^{2i\theta})(1-\bar{\gamma}_2 e^{2i\theta})} \tag{10}$$

$$\gamma_1 = \frac{\alpha_1 - 1}{\alpha_1 + 1}, \quad |\gamma_1| < 1, \qquad \gamma_2 = \frac{\alpha_2 - 1}{\alpha_2 + 1}, \quad |\gamma_2| < 1 \tag{11a}$$

$$\alpha_1^2\alpha_2^2 = \frac{S_{11}}{S_{22}}, \qquad \alpha_1^2 + \alpha_2^2 = \frac{2S_{12} + S_{33}}{S_{22}} \tag{11b}$$

where $\bar{V}(\bar{z})$ and $\bar{W}(\bar{z})$ are the conjugates of the complex potentials $V(z)$ and $W(z)$, respectively.

The displacement, in terms of complex variables, is $D = u_x + iu_y$, where $u_x$ is the displacement in $x$-direction and $u_y$ is the displacement in $y$-direction. At the circumference of the tunnel $r = |z| = a$, it has been shown by Green and Zerna[12] that the displacement $D$ may be expressed in terms of two complex functions $f(z)$ and $g(z)$ and their conjugates $\bar{f}(\bar{z})$ and $\bar{g}(\bar{z})$ as

$$D = \delta_1 f(z) + \bar{\rho}_1 \bar{f}(\bar{z}) + \delta_2 g(z) + \bar{\rho}_2 \bar{g}(\bar{z}) \tag{12}$$

The functions $f(z)$ and $g(z)$ are defined by their derivatives $f'(z)$ and $g'(z)$ which are functions of the complex potentials $W(z)$ and $V(z)$, which have been determined

$$f'(z) = \frac{\left(1 + \gamma_2 \dfrac{a^2}{z^2}\right)V(z) + \left(1 - \gamma_2 \dfrac{a^2}{z^2}\right)W(z)}{4(\gamma_1 - \gamma_2)} \tag{13a}$$

$$g'(z) = -\frac{\left(1 + \gamma_1 \dfrac{a^2}{z^2}\right)V(z) + \left(1 - \gamma_1 \dfrac{a^2}{z^2}\right)W(z)}{4(\gamma_1 - \gamma_2)} \tag{13b}$$

where $\delta_1$, $\delta_2$, $\rho_1$, and $\rho_2$ are functions of the elastic constants and given by

$$\begin{aligned} \delta_1 &= (1+\gamma_1)\beta_2 - (1-\gamma_1)\beta_1, \qquad & \delta_2 &= (1+\gamma_2)\beta_1 - (1-\gamma_2)\beta_2 \\ \rho_1 &= (1+\gamma_1)\beta_2 + (1-\gamma_1)\beta_1, \qquad & \rho_2 &= (1+\gamma_2)\beta_1 + (1-\gamma_2)\beta_2 \end{aligned} \tag{14}$$

In general, two cases arise:

(a) $\bar{\gamma}_1 = \gamma_1, \quad \bar{\gamma}_2 = \gamma_2$
so that $\gamma_1$ and $\gamma_2$ are real. In this case, $\bar{\rho}_1 = \rho_1$ and $\bar{\rho}_2 = \rho_2$ (i.e. $\rho_1$ and $\rho_2$ are real)
(b) $\bar{\gamma}_2 = \gamma_1, \quad \bar{\rho}_1 = \rho_2$
so that $\gamma_1$ and $\gamma_2$ are complex conjugates. In this case, $\rho_2 = \bar{\rho}_1$ and $\rho_1 = \bar{\rho}_2$ (i.e. $\rho_1$ and $\rho_2$ are complex conjugates).

For isotropic media, $\alpha_1 = \alpha_2 = 1$, therefore,

$$\gamma_1 = \gamma_2 = \delta_1 = \delta_2 \to 0, \quad \rho_1 = \rho_2 \to \frac{-2(1+v)}{E} \tag{16}$$

Equation (10), with the aid of equations (7) and (11), is the formal solution for the tangential stress, while equation (12), with the aid of equations (7), (13)–(15), is the formal solution for the radial displacement. In the following sections, these formal solutions will be reduced to engineering quantities.

## CLOSED-FORM SOLUTIONS OF THE STRESSES AND DISPLACEMENTS

Consider a tunnel driven in a cross-anisotropic rockmass acting upon by the initial stresses $P_v$ and $P_h$ as shown in Figure 1. Utilizing the theory discussed in the previous section, the closed-form solution for the tangential stresses along the circumference of the circular tunnel is derived as

$$\sigma_\theta = \frac{2 + 2(\gamma_1 + \gamma_2)^2 - 2\gamma_1^2\gamma_2^2 - 4(\gamma_1 + \gamma_2)\cos 2\theta}{(1 + \gamma_1^2 - 2\gamma_1 \cos 2\theta)(1 + \gamma_2^2 - 2\gamma_2 \cos 2\theta)} P_0 + \frac{4(\gamma_1 + \gamma_2) - 4(1 + \gamma_1\gamma_2)\cos 2\theta}{(1 + \gamma_1^2 - 2\gamma_1 \cos 2\theta)(1 + \gamma_2^2 - 2\gamma_2 \cos 2\theta)} Q_0 \tag{17}$$

where $P_0$ is the hydrostatic initial stress component $((P_h + P_v)/2)$ and $Q_0$ is the deviatoric initial stress component $((P_h - P_v)/2)$. The solutions derived for the radial displacements ($u_a$) and the tangential displacements ($u_\theta$), respectively, are

$$u_a = \frac{a}{2(\gamma_1 - \gamma_2)} \{P_0(\gamma_2\rho_1 - \gamma_1\rho_2) + Q_0(\rho_1 - \rho_2) + [P_0(\gamma_2\delta_1 - \gamma_1\delta_2) + Q_0(\delta_1 - \delta_2)]\cos 2\theta\} \tag{18}$$

$$u_\theta = \frac{a}{2(\gamma_1 - \gamma_2)} [P_0(\gamma_1\delta_2 - \gamma_2\delta_1) + Q_0(\delta_2 - \delta_1)]\sin 2\theta \tag{19}$$

*Case of pressure tunnel*

The tangential stresses at the circumference of a pressure tunnel of an internal pressure ($P_i$) in a cross-anisotropic rock is given by equation (17) with the substitution of $(P_0 - P_i)$ in place of $(P_0)$. Therefore, the solution for the tangential stresses in case of a pressure tunnel is

$$\sigma_\theta = \frac{2 + 2(\gamma_1 + \gamma_2)^2 - 2\gamma_1^2\gamma_2^2 - 4(\gamma_1 + \gamma_2)\cos 2\theta}{(1 + \gamma_1^2 - 2\gamma_1 \cos 2\theta)(1 + \gamma_2^2 - 2\gamma_2 \cos 2\theta)} (P_0 - P_i) + \frac{4(\gamma_1 + \gamma_2) - 4(1 + \gamma_1\gamma_2)\cos 2\theta}{(1 + \gamma_1^2 - 2\gamma_1 \cos 2\theta)(1 + \gamma_2^2 - 2\gamma_2 \cos 2\theta)} Q_0 \tag{20}$$

The radial and tangential displacements resulting from the internal pressure are given by equations (18) and (19), respectively, with the substitution of $-P_i$ in place of $P_0$ and $Q_0 = 0$. Therefore, the radial and tangential displacements (due to internal pressure only) in the case of

 

pressure tunnel, respectively, are

$$u_a = \frac{aP_i}{2(\gamma_1 - \gamma_2)} \{(\gamma_1 \rho_2 - \gamma_2 \rho_1) + (\gamma_1 \delta_2 - \gamma_2 \delta_1)\cos 2\theta\} \tag{21}$$

$$u_\theta = \frac{aP_i}{2(\gamma_1 - \gamma_2)} (\gamma_2 \delta_1 - \gamma_1 \delta_2)\sin 2\theta \tag{22}$$

## COMPARISON TO SOME CASES OF KNOWN SOLUTIONS

### *(a) Green and Taylor solution for stresses*

Closed-form solutions for displacements on the circumference of an excavated tunnel in an elastic cross-anisotropic medium resulting from the relief of the *in situ* stress have not been obtained in the literature. However, the stress distribution around a pre-existing hole in an infinite plate loaded at infinity from one direction is known.[8,9] For an elastic medium, the stress distribution from a pre-existing hole or from stress relief at the hole boundary in pre-loaded medium should be the same. Therefore, the closed-form solutions for stresses, when $P_v = 0$, should reduce to that given by Green and Taylor.

Under the existence of a uniform pressure $T$ parallel to the horizontal direction ($x$-axis) the tangential stress $\sigma_\theta$ at the hole boundary is given by Green and Taylor as

$$\sigma_\theta = \frac{T(1+\gamma_1)(1+\gamma_2)(1+\gamma_1+\gamma_2-\gamma_1\gamma_2 - 2\cos 2\theta)}{(1+\gamma_1^2 - 2\gamma_1 \cos 2\theta)(1+\gamma_2^2 - 2\gamma_2\cos 2\theta)} \tag{23}$$

This corresponds to the case of: $P_v = 0$ and $P_h = T$. Therefore, $P_0 = Q_0 = T/2$. Substituting for the values of $P_0$ and $Q_0$ in equation (17), an identical solution as given by Green and Taylor is obtained (equation (23)).

### *(b) Case of isotropic medium*

The solutions for the stresses and displacements for the case of a tunnel excavated in elastic isotropic rocks are well known (see e.g. References 6 and 7). The tangential stresses and the radial and tangential displacements at the circumference of the tunnel wall, in terms of the hydrostatic and deviatoric components of initial stresses, are, respectively,

$$\sigma_\theta = 2P_0 - 4Q_0 \cos 2\theta \tag{24}$$

$$u_a = \frac{a(1+v)}{E} [P_0 + (3-4v)Q_0 \cos 2\theta] \tag{25}$$

$$u_\theta = -\frac{a(1+v)}{E} Q_0 (3-4v)\sin 2\theta \tag{26}$$

Taking the limit of $\sigma_\theta$, $u_a$, and $u_\theta$ in equations (17)–(19), respectively, as

$$\gamma_1 = \gamma_2 = \delta_1 = \delta_2 \to 0 \quad \text{and} \quad \rho_1 = \rho_2 \to \frac{-2(1+v)}{E},$$

Equations (17)–(19) are reduced to equations (24)–(26), respectively.

 

## EXAMPLE OF DESIGN ANALYSIS

For the design of Sir Adam Beck Niagara Generating Station (SABNGS) No. 3 project (Niagara Falls, Ontario), twin tunnels, each of approximately 13 m diameter and 10 km long are required.[13] The tunnels will be excavated in Queenston Shale at depths of 150 to 200 m. The five independent parameters required to describe the deformation behaviour of Queenston Shale, using the theory of elastic orthotropic (cross-anisotropic) material, were determined using the method presented by Lo and Hori.[14] Typical values for the parameters obtained are: $E_h = 15{\cdot}8$ GPa, $E_v = 10{\cdot}5$ GPa, $G_{vh} = 3{\cdot}95$ GPa, $\upsilon_{vh} = 0{\cdot}3$, and $\upsilon_h = 0{\cdot}3$, implying, $E_h/E_v = 1{\cdot}5$, $E_h/G_{vh} = 4$, and $\upsilon_{hv} = 0{\cdot}44$. The values of $E_v$ and $\upsilon_{vh}$ are usually taken to represent the elastic parameters of the material if the theory of elastic isotropic material is adopted.

The initial stresses at a typical cross-section of the tunnels at depth 200 m from the ground surface are: $P_v = 5{\cdot}2$ MPa and $P_h = 21$ MPa, giving: $K_0 = 4$, $P_0 = 13{\cdot}1$ MPa, and $Q_0 = 7{\cdot}9$ MPa.

Using the values of the five elastic parameterss discussed above and equations (2) and (11), the following values of $\gamma_1$ and $\gamma_2$ are obtained:

$$\gamma_1 = 0{\cdot}1432, \qquad \gamma_2 = -0{\cdot}2296$$

The values of $\delta_1$, $\delta_2$, $\rho_1$, and $\rho_2$ are obtained, using equation (14), as

$$\delta_1 = 7{\cdot}813 \times 10^{-5}, \quad \delta_2 = -5{\cdot}625 \times 10^{-5}$$

$$\rho_1 = -2{\cdot}368 \times 10^{-4}, \quad \rho_2 = -2{\cdot}269 \times 10^{-4}$$

The tangential stresses and radial displacements are obtained by substituting the values of $P_0$, $Q_0$, $\gamma_1$, $\gamma_2$, $\delta_1$, $\delta_2$, $\rho_1$, and $\rho_2$ in equations (17) and (18), respectively. For comparison, the corresponding values of stresses and displacements for isotropic rock condition are also calculated using equations (24) and (25).

Figure 2 shows the distributions of the tangential stresses and radial displacements at the circumference of the tunnels for the two cases considered (isotropic and cross-anisotropic). A summary of the stresses and displacements obtained at the crown and the springline is given in Table I. The differences between the two cases are represented as percentages of the results obtained for the cross-anisotropic case. From Figure 2 and Table I, the following observations may be made.

(a) Compressive stress concentration at the crown-invert with a stress concentration factor of 3 is predicted by the anisotropic theory. The assumption of isotropy leads to a small difference of only 9 per cent.
(b) Both isotropic and anisotropic theories predict tangential tensile stress at the springline. However, consideration of anisotropy leads to significantly lower values, the difference in magnitude being as much as 170 per cent.
(c) Both theories predict inward displacements at the springline; consideration of anisotropy gives lower predicted values by 27 per cent.
(d) At the crown-invert, the predicted direction of elastic displacements depends on the theory used, although the magnitudes of displacements are small.

These observations are significant in the interpretation of results of field monitoring of stresses and displacements in test adits and during construction of tunnels.

 

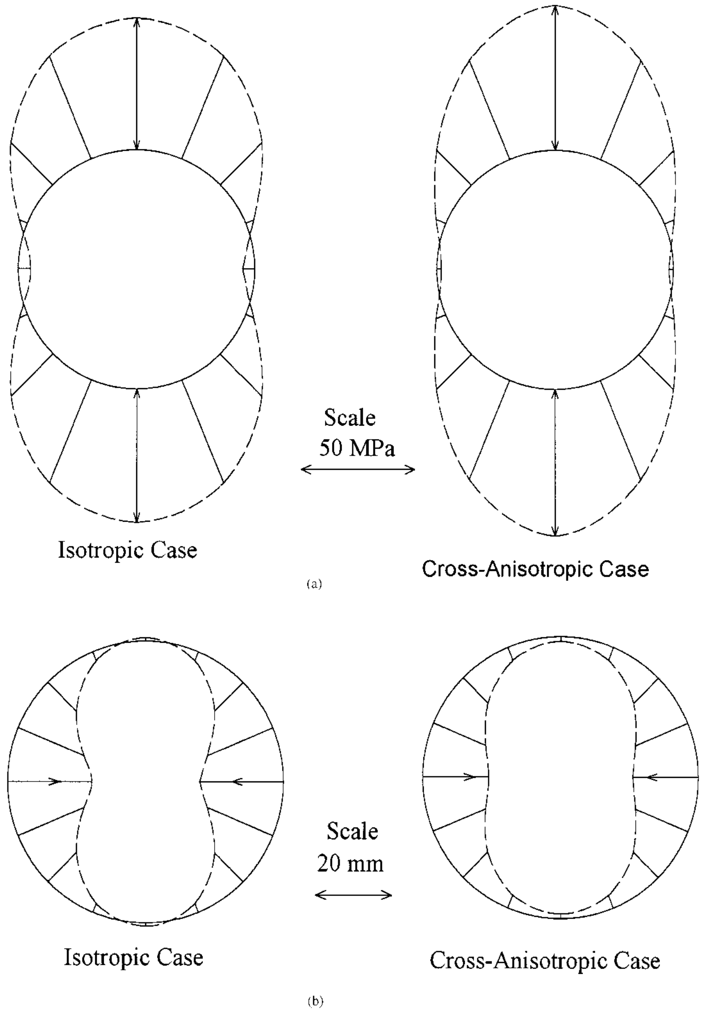


Figure 2. (a) Tangential stresses and (b) radial displacements at the circumference of the proposed tunnels in Queenston Shale for the SABNGS No. 3 project

Table I. Stresses and displacements at the circumference of a tunnel driven in Queenston Shale

| Stresses (MPa) | | | | | | Displacements (mm) | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| Springeline | | | Crown | | | Springline | | | Crown | | |
| Iso-tropic | Aniso-troppic | % diff | Iso-tropic | Aniso-tropic | % diff | Iso-tropic | Aniso-tropic | % diff | Iso-tropic | Aniso-tropic | % diff |
| − 5·4 | − 2·0 | 170 | 57·8 | 63·4 | 8·8 | 21·9 | 17·2 | 27 | − 0·9 | 1·2 | 178 |

## EFFECTS OF ANISOTROPIC ELASTIC PARAMETERS AND $K_0$ ON STRESSES AND DISPLACEMENTS

The five elastic parameters characterizing the deformation behaviour of the rock ($E_v$, $E_h$, $v_h$, $v_{vh}$, and $G_{vh}$) can be determined using the method described by Lo and Hori.[14] Experience gained from testing of different shaly rocks from Southern Ontario showed that the elastic paramters generally vary within the following ranges:

$$v_{vh}\text{: } 0{\cdot}2 \text{ to } 0{\cdot}4, \quad v_h\text{: } 0{\cdot}2 \text{ to } 0{\cdot}4, \quad E_h/E_v\text{: } 1 \text{ to } 2, \quad E_h/G_{vh}\text{: } 3 \text{ to } 10$$

*In situ* stress measurements in the same rock formations showed that the stress ratios ($K_0$) generally lie in the range of 3–30.[15] These ranges of values are used to study the effect of $v_h$, $v_{vh}$, $E_h/G_{vh}$ and $K_0$ on the tangential stresses and radial displacements. For the presentation of results, stresses and displacements are expressed in dimensionless forms. The tangential stress $\sigma_\theta$ is normalized by the initial stress hydrostatic component $P_0$, and the radial displacement $u_a$ is expressed by the dimensionless displacement $\Omega = u_a E_h/aP_0$. The results are shown in Figures 3 and 4.

### *(a) Effect of $v_h$*

Referring to Figures 3(a) and 4(a), it may be observed that the effects of $v_h$ on both the tangential stresses and radial displacements are negligible.

### *(b) Effect of $v_{vh}$*

The effect of $v_{vh}$ on the tangential stress and radial displacement is shown on Figures 3(b) and 4(b), respectively. It may be observed from Figure 3(b) that the effect of $v_{vh}$ on the tangential stresses is small. The effects of $v_{vh}$ on the displacements (Figure 4(b)), however, show some interesting and significant features. Starting from the springline, the effect of $v_{vh}$ progressively increases towards the crown-invert. At the crown-invert, an increase of $v_{vh}$ from 0·2 to 0·4 leads to a change of $\Omega$ from $-0{\cdot}1$ to 0·5. Therefore, not only the magnitude of $\Omega$ is affected, but also the direction of displacement is altered, for the case considered.

### *(c) Effect of $E_h/G_{vh}$*

Figures 3(c) and 4(c) show the effect of $E_h/G_{vh}$ on the tangential stresses and radial displacements, respectively. It may be observed from Figure 3(c) that as the value of $E_h/G_{vh}$ increases, the compressive tangential stresses at both the springline and the crown increase, with higher stress concentration at the crown. It may be seen from Figure 4(c) that the radial displacements at both the crown and the springline are highly sensitive to the value of $E_h/G_{vh}$, increasing with $E_h/G_{vh}$. Therefore, $E_h/G_{vh}$ is an important deformation parameter to consider in predicting displacements.

### *(d) Effect of $K_0$*

The effects of $K_0$ on the tangential stress and radial displacement are shown on Figures 3(d) and 4(d), respectively. It may be observed from Figure 3(d) that the compressive stress concentra-

 

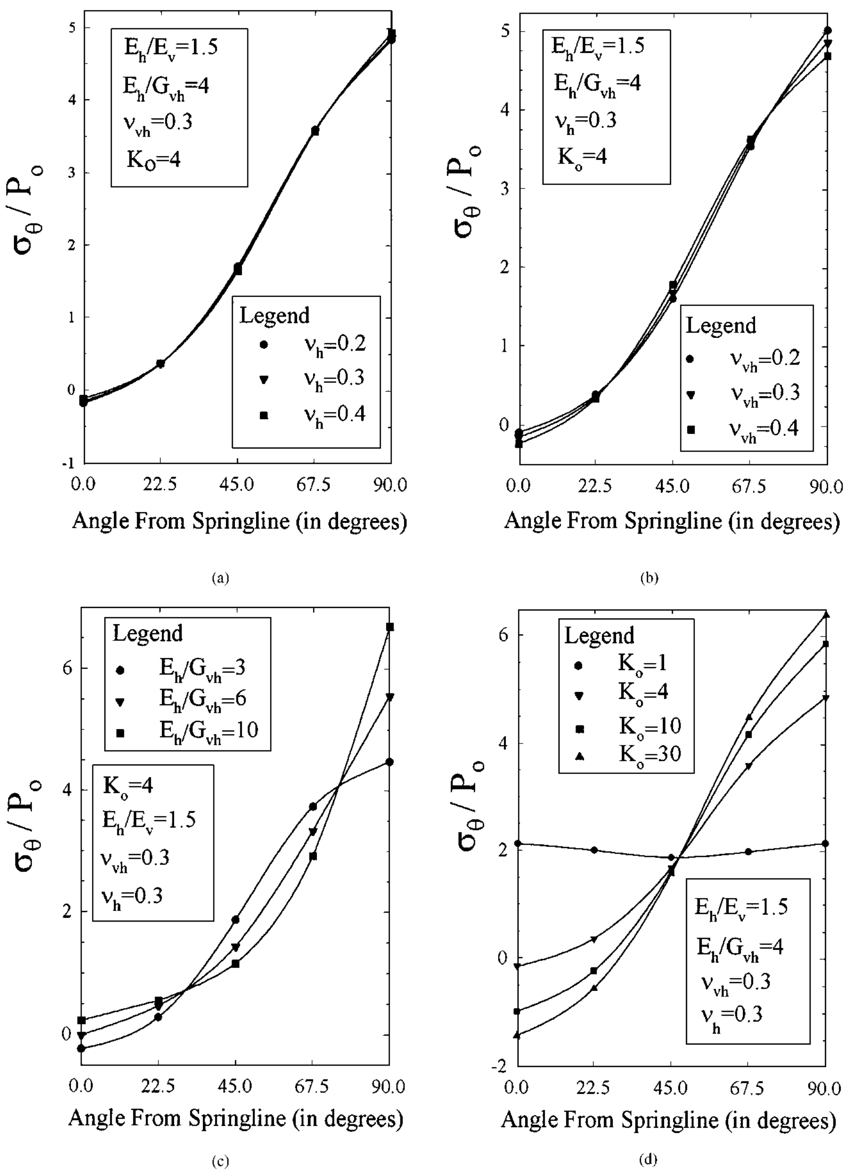


Figure 3. Effect of: (a) $v_h$, (b) $v_{vh}$, (c) $E_h/G_{vh}$, (d) $K_0$ on the distribution of tangential stress at the circumference of a tunnel driven in a cross-anisotropic medium

tion increases at the crown as the value of $K_0$ increases. On the other hand, the stress at the springline decreases and becomes tensile as the value of $K_0$ increases.

It is obvious from Figure 4(d) that the dimensionless displacement factor ($\Omega$) at the springline increases (indicating greater inward displacement) as the value of $K_0$ increases. At the

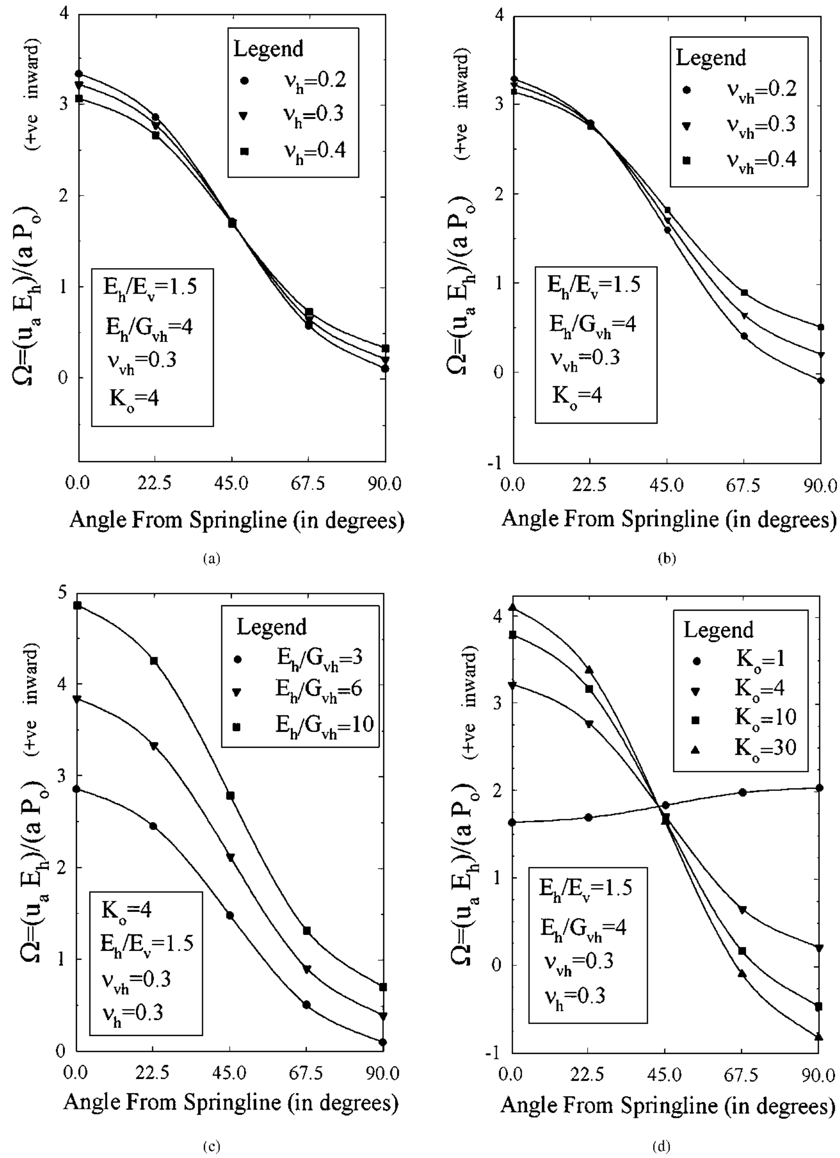


Figure 4. Effect of: (a) $\upsilon_h$, (b) $\upsilon_{vh}$, (c) $E_h/G_{vh}$, (d) $K_0$ on the distribution of radial displacement at the circumference of a tunnel driven in a cross-anisotropic medium

crown-invert, the value of $\Omega$ decreases and then changes to outward displacement as $K_0$ increases.

Under isotropic stress condition ($K_0 = 1$), it is well known that the assumption of isotropy will lead to uniform stresses and displacements. It is interesting to observe from Figures 3(d) and 4(d) that both the stress and displacement are non-uniform for $K_0 = 1$. This is a direct result of

 

material anisotropy, which induces shear stresses and distortion even under external uniform pressure.

It may be concluded from the previous discussion that the elastic parameters controlling the degree of anisotropy of the rock have significant effects on stresses and displacements around the tunnel, and should be taken into consideration in design analysis.

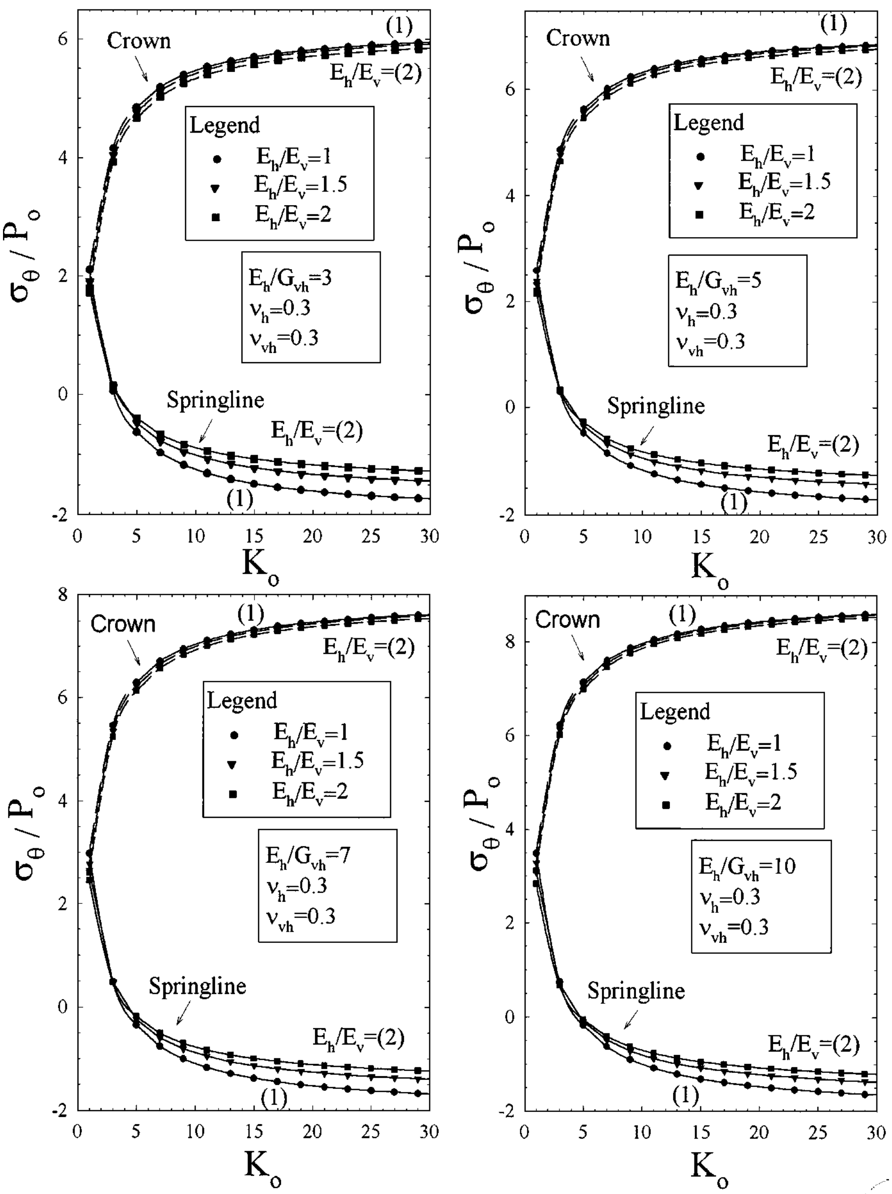


Figure 5. Tangential stresses at the circumference of a tunnel driven in a cross-anisotropic medium ($\upsilon_{vh} = 0{\cdot}3$)

## CHARTS FOR DETERMINATION OF STRESSES AND DISPLACEMENTS

To facilitate calculation of stresses, Figure 5 shows the relationships of $\sigma_\theta/P_0$ with $K_0$, for different values of $E_h/G_{vh}$ and $E_h/E_v$. The results are for $\upsilon_{vh} = 0{\cdot}3$.

From Figure 5, it may be seen that the effect of $E_h/E_v$ on stresses at the crown is negligible, for any value of $K_0$. At the springline, the magnitude of stress decreases slightly as $E_h/E_v$ increases from 1 to 2.

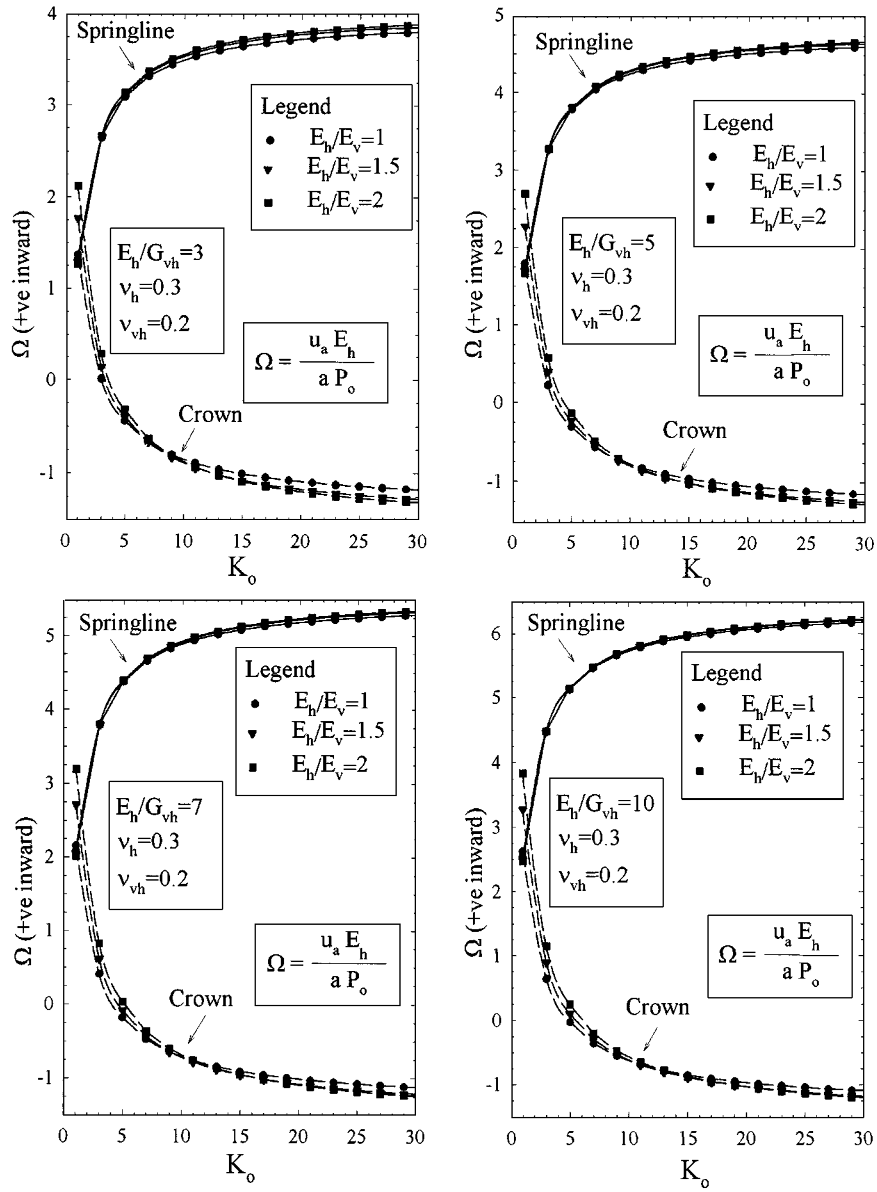


Figure 6. Radial displacements at the circumference of a tunnel driven in a cross-anisotropic medium ($\upsilon_{vh} = 0{\cdot}2$)

 

The relationships for the dimensionless displacements $\Omega$ with $K_0$ for $v_{vh} = 0{\cdot}2$, $0{\cdot}3$ and $0{\cdot}4$ are shown in Figures 6, 7, and 8, respectively. It is obvious from these figures that the radial displacement at the springline is insensitive to the value of $E_h/E_v$. On the other hand, the magnitude and direction of the radial displacement at the crown-invert is highly sensitive to

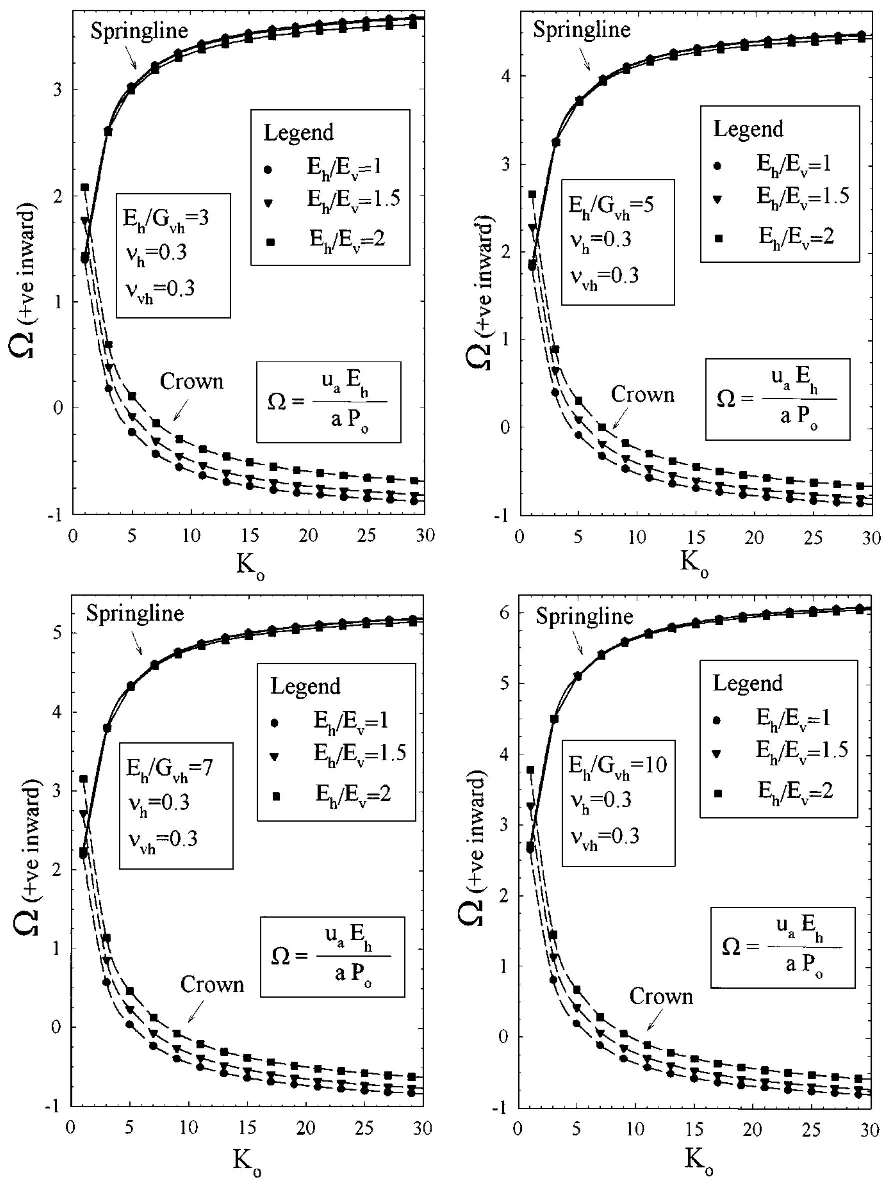


Figure 7. Radial displacements at the circumference of a tunnel driven in a cross-anisotropic medium ($v_{vh} = 0{\cdot}3$)

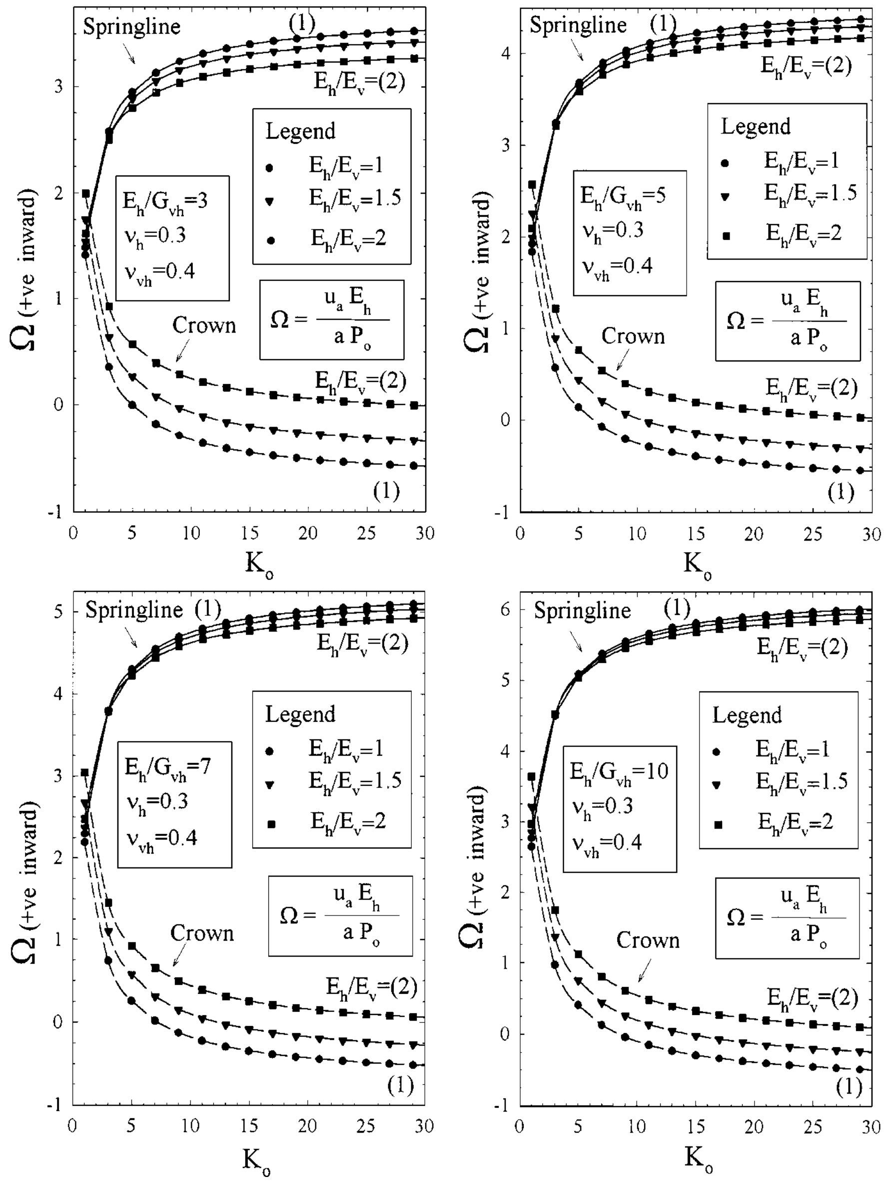


Figure 8. Radial displacements at the circumference of a tunnel driven in a cross-anisotropic medium ($v_{vh}$ = 0·4)

$E_h/E_v$, specially for values of $v_{vh}$ higher than 0·2. For example, for the case of $v_{vh}$ = 0·4 (Figure 8), as $E_h/E_v$ increases, the displacement changes direction from outward to inward displacement generally for values of $K_0$ higher than 6. This is an important factor that should be taken into consideration in prediction of displacements.

 

## CONCLUSIONS

The stresses and displacements around an unlined tunnels driven in cross-anisotropic rocks have been investigated. Closed-form solutions for the stresses and displacements are derived utilizing the stress potentials developed by Green and Zerna.[12]

The effects of anisotropy of the initial stress system and the elastic parameters on the stresses and displacements are studied. From the results of this investigation, the following conclusions may be drawn:

(a) The effects of $v_h$ on both of the tangential stress and radial displacements are negligible.
(b) At the crown-invert, the magnitude and direction of the radial displacement are sensitive to the value of $v_{vh}$.
(c) The compressive stress at the crown increases with increasing $E_h/G_{vh}$. The radial displacements around the tunnel are highly sensitive to the value of $E_h/G_{vh}$, increasing substantially as $E_h/G_{vh}$ increases.
(d) As the value of $K_0$ increases, the compressive stress concentration at the crown-invert increases and the stress at the springline decreases and changes to tensile. The dimensionless displacement factor ($\Omega$) at the springline increases (indicating greater inward displacement) as the value of $K_0$ increases. In contrast, the value of $\Omega$ at the crown decreases and then changes to outward displacement as $K_0$ increases.
(e) The radial displacement at the springline is insensitive to $E_h/E_v$ for any value of $K_0$. On the other hand, the magnitude and direction of the radial displacement at the crown-invert is highly sensitive to the value of $E_h/E_v$, specially for values of $v_{vh}$ greater than 0·2.

For convenience of application, charts (Figures 5–8) have been prepared for the determination of stresses and displacements for given values of initial stresses and the elastic parameters.

It is believed that the results presented would be useful not only for ready determination of stresses and displacements for design considerations, but also for the interpretation of results of field monitoring in test adits and during construction.

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